

A REMARK ON DEGENERATE SINGULARITY IN THREE DIMENSIONAL RICCI FLOW

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ABSTRACT. We show that a rescale limit at any degenerate singularity of Ricci flow in dimension 3 is a steady gradient soliton. In particular, we give a geometric description of type I and type II singularities.

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Degenerate singularity of the Ricci flow

$$(0.1) \quad \frac{d}{dt}g(t) = -2 \operatorname{Ric}(g(t))$$

was introduced in Hamilton's paper [10]. In that paper, Hamilton first described the *nondegenerate neck-pinching*. Roughly speaking, one starts the Ricci flow on a dumbbell shaped 3-manifold, with the neck diffeomorphic to $S^2 \times [-1, 1]$. It is expected that the neck shrinks in the S^2 direction, where the curvature is very positive, and, at the same time, stays relatively stationary in the \mathbb{R} direction, where the curvature is slightly negative. After some time, the neck pinches off and forms a singularity. One step further, Hamilton purposed the notion of *degenerate neck-pinching*: reduce the left half of the dumbbell into a critical size and then start the Ricci flow. It is expected that after some time, all of the left half of the dumbbell pinches off, and forms a singularity like a horn growing out of the (remaining) right half of the dumbbell. See [10] for further descriptions and some very inspiring pictures.

In this paper, we prove that in dimension 3, a rescale limit of a degenerate singularity of Ricci flow is a steady gradient soliton, see Theorem 3.22. The precise definition of degenerate singularity, given in Definition 1.7, is based on Perelman's notion of *canonical neighborhood*. Our definition is a geometric one that reflects Hamilton's original picture in [10]. On the other hand, as we will see later in this paper, this geometric definition is equivalent to that the singularity being of type II.

For previous works on neck-pinching, see [1], [2], and the book [5]. In the book [4], there is a detailed treatment of nondegenerate neck-pinching in chapter 2, and a discussion of degenerate neck pinching in page 62-66.

We start by reviewing some of Perelman's results in [14], [15]; for more details, see [3], [12] and [13]. In section 2 we use an estimate on Perelman's l functional to rule out noncompact ancient solutions with positive curvature that develops a type I singularity. Therefore a rescale limit of a degenerate singularity is either an eternal solution or an ancient solution that develops a type II singularity. In both cases, we need to take a further rescale limit in *forward time*; we treat certain issues related to this in Section 3. Then we use a theorem of Hamilton [9] to conclude that the final rescale limit is a steady soliton. Our arguments are similar to Perelman's compactness/convergence methods that were used extensively in his papers [14], [15]; for reader's convenience we will give a detailed account.

Recently it comes to our attention that Gu and Zhu [7] proved the existence of type II singularity; they used Perelman's l functional argument to detect type II singularity in the radial symmetric case. See also a very recent paper [6].

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1. NOTATIONS AND DEFINITION

All manifolds we consider in this paper are of dimension 3. We use R , often with two variables x and t , to denote the scalar curvature. Rm denotes the full curvature tensor. The Hamilton-Ivey pinching inequality (see [11]) says, if in the beginning the curvature is bounded from below by -1 , then we have

$$\text{Rm} \geq -\phi(|R|),$$

where ϕ is a nonnegative function so that $\lim_{r \rightarrow 0} \phi(r)/r = 0$. In particular, when R is large, the full curvature is dominated by the scalar curvature R .

We follow some notations of Perelman [14], [15]. Assume $g(t)$ is a family of metrics on a manifold M that evolve under the Ricci flow. $B(x, t, r)$ denotes the metric ball centered at x , of radius r with respect to the metric $g(t)$. Then one defines the *parabolic neighborhood*

$$(1.1) \quad P(x, t, r, -\Delta t) = B(x, t, r) \times [t - \Delta t, t].$$

When we say two sets (U_1, p_1, t_1) and (U_2, p_2, t_2) are ϵ -close, typically we first do rescales on U_1 and on U_2 so that $R(p_1, t_1) = R(p_2, t_2) = 1$, then ϵ -close means these rescaled sets are ϵ -close under C^5 topology. There is a similar notion of ϵ -close between two parabolic neighborhoods.

Since we study only finite time singularities, by 7.3 of [14], all solutions are *noncollapsing*. In particular, when a sequence of parabolic neighborhoods admit a uniform curvature bound, we can take pointed limit over a subsequence.

An *ancient solution* is a solution that exists on the time interval $(-\infty, T)$ for some $T \in \mathbb{R}$. The cylinder $S^2 \times \mathbb{R}$, with the S^2 direction evolving under the Ricci flow, is an important example. A set (Z, z) evolving under the Ricci flow over time $[-t, 0]$ is called a *strong ϵ -neck*, if after rescale the metrics by $R(z, 0)$, Z is ϵ -close to an evolving cylinder of length ϵ^{-1} from time $-\epsilon^{-1}$ to 0.

Definition 1.2. The **caliber** of a cylinder $Y = S^2 \times \mathbb{R}$ is R^{-1} , where R is the scalar curvature of Y .

Clearly the caliber of a cylinder Y is just half of the square of its radius. The following is obvious:

Lemma 1.3. Start the Ricci flow at time 0 on Y , then the caliber of Y equals to the time it takes for the cylinder to go singular (i.e. shrink into the real line \mathbb{R}).

Proposition 1.4 (Perelman). Given $\epsilon > 0$. Assume the initial metric $g(0)$ satisfies the curvature bound $|\text{Rm}(g(0))| \leq 1$, and for all $p \in M$, $\text{Vol}_{g(0)}(B_1(p)) \geq 10^{-1}$.

Then there exists $r_0 > 0$, so that whenever $R(x_1, t_1) > r_0^{-2}$, the neighborhood $P(x_1, t_1, R(x, t)^{1/2}\epsilon^{-1}, -\epsilon^{-1}R(x, t))$, under the rescaled metric $R(x_1, t_1)g(t)$, is ϵ -close to a parabolic neighborhood in one of the following:

- i). A space form with positive curvature evolving under the Ricci flow,
- ii). The cylinder $S^2 \times \mathbb{R}$ (or $S^2 \times \mathbb{R}/\mathbb{Z}_2$) evolving under the Ricci flow,
- iii). A compact ancient solution with strictly positive, nonconstant (at each time slice) curvature that is diffeomorphic to S^3 or \mathbb{RP}^3 ,
- iv). A noncompact ancient solution to the Ricci flow with strictly positive curvature.

For a proof, see theorem 12.1 of [14], together with section 1 of [15]. The possibilities i), ii), iii), iv) above give a rough classification of noncollapsing ancient solutions of nonnegative curvature in dimension 3. The parabolic neighborhood P above is called a **canonical neighborhood**.

Both of the ancient solutions iii) (in sufficiently ancient time), and iv) (in all time), contain a piece of evolving cylinders; see [14] sections 11, 12 and especially 1.4 of [15]. To emphasize the difference between cases ii), iii), iv), remember

Proposition 1.5 (Perelman). Assume X is an ancient, noncollapsing 3-dimensional ancient solution with nonnegative, nonconstant (at each time slice) curvature. Then at each time (e.g. $t = 0$), X can be decomposed into two parts, X_C and X_T ; either of them can be empty. X_T is connected; X_C has at most two connect components and is connected when M is noncompact. The boundary components (if any) of X_C and X_T are all diffeomorphic to S^2 .

Each connected component of X_C is compact, for all p, q in the same connect component of X_C , we have

$$(1.6) \quad R(p, 0) \leq A(\epsilon)R(q, 0), \quad \text{Diam } X_C \leq D(\epsilon)R(p, 0)^{-1/2}.$$

Every point in the second part, X_T , is the center of a strong ϵ -neck.

Moreover, when X_C is connected, then it is diffeomorphic to either S^3 , or RP^3 , or the 3-dimension ball B^3 . When X_C is not connected and therefore has two components, then one component of X_C is diffeomorphic to B^3 , the other is diffeomorphic to either B^3 or $RP^3 - B^3$.

Most of the above is proved in section 11 (especially 11.8) of [14], the last part concerning the topology of X_C can be found in section 1 of [15]; the proof uses a compactness argument (go to ancient time) involving the soul theorem, see for example chapter 9 of [13], [12] and [3].

Roughly speaking, the above Proposition says the following. Assume X is not a space form, then X can be decomposed into a “tube part” X_T and a “cap part” X_C . More precisely, when X is noncompact, then either X is just the tube, or it is an approximate tube being connected, through one S^2 boundary, to a cap that is diffeomorphic to B^3 or $RP^3 - B^3$. The later case is isometric to $S^2 \times \mathbb{R}/\mathbb{Z}_2$.

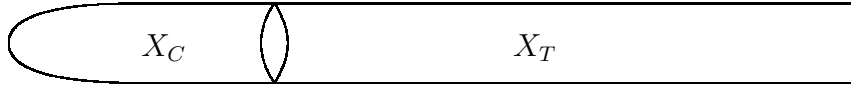


Figure I.

Figure I shows the cap-tube decomposition of the ancient solution case iv) in Proposition 1.4. We can take the convention that, if the canonical neighborhood in Proposition 1.4 fall into case iv) *and* the canonical neighborhood lies in the X_T part, then we shall classify this neighborhood into case ii). In another word, when the canonical neighborhood is of case iv), then the neighborhood contains X_C .

When X is compact, we have two cases. If X is not “long enough”, then X itself is X_C , which is diffeomorphic to S^3 or RP^3 ; if X is “long enough”, then X comes from a bounded tube X_T being capped at its two ends by two caps X_C : at one of these cap must be B^3 , the other may be either B^3 or $RP^3 - B^3$.

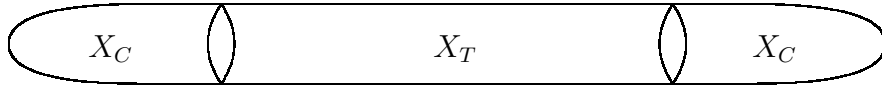


Figure II.

Figure II is a picture of the ancient solution iii in Proposition 1.4; this is the “long” case so that the tube part X_T is nonempty.¹ In this case, if the canonical neighborhood

¹ As we will see in the proof of Theorem 3.22, case iii) can be ignored in our study.

falls on X_T , or, on a component of X_C that is diffeomorphic to $RP^3 - B^3$, then we classify it into case ii of Proposition 1.4; if it falls on a component of X_C that is diffeomorphic to B^3 , by a compactness argument we classify it into iv) of Proposition 1.4.

In particular, we can choose ϵ so that the options i), ii), iii), iv) of Proposition 1.4 are **mutually exclusive**. With these information, we give a working definition of *degenerate singularity* of Ricci flow in dimension 3:

Definition 1.7. *Assume M is compact and $g(t)$ is a solution to the Ricci flow as in Proposition 1.4 that exists on the time interval $[0, T)$ with $T < \infty$. Assume there is a sequence of points (x_i, t_i) so that $\lim_{i \rightarrow \infty} t_i = T$ and $\lim_{i \rightarrow \infty} R(x_i, t_i) = \infty$; moreover for sufficiently large i the canonical neighborhood of (x_i, t_i) is of case iv) in Proposition 1.4. Then we say a degenerate singularity happens at time T .*

A glance at Hamilton's picture in [10] suggests that one might also include case ii) in Proposition 1.4, when the canonical neighborhood is $S^2 \times \mathbb{R}/\mathbb{Z}_2$; we will discuss this possibility in the end of this paper.

It is possible that the solution goes singular *everywhere*, i.e. scalar curvature goes to infinity everywhere as $t \rightarrow T^-$. At this moment we don't know a compact example that extincts everywhere while developing a degenerate singularity. On the other hand, Perelman's *standard solution* in Section 2 of [15] is a noncompact example. If the solution extincts everywhere at T , then the topology of M is quite simple, see sections 3 and 4 of [15].

There is an important gradient estimate for scalar curvature, see (1.3) of [15]:

Proposition 1.8 (Perelman). *Let $g(t)$ be a solution to the Ricci flow as in Proposition 1.4 and r_0 be the canonical neighborhood parameter. Then whenever $R(x, t) > r_0^{-2}$, we have*

$$(1.9) \quad |\nabla R(x, t)| \leq C_1 R^{3/2}, \quad |\partial_t R|(x, t) \leq C_2 R^2.$$

The next is a very useful locally splitting theorem:

Proposition 1.10 (Perelman). *Given any ϵ , there exists $\eta > 0$ so that the following is true:*

Assume X is a noncollapsing, noncompact, ancient solution of nonnegative, bounded curvature defined for time $t \in (-\infty, 0]$. Assume γ is a minimal geodesic segment at time 0, with two end points p_1 and p_2 , and p is a point on γ . Assume

$$(1.11) \quad R(p_1, 0) > \eta^{-1}, \quad R(p, 0) = 1, \quad R(p_2, 0) < \eta.$$

Then $d(p_1, p) > \epsilon^{-1}$, $d(p, p_2) > \epsilon^{-1}$, and $B(p, 0, \epsilon^{-1})$ is ϵ -close to a subset in a cylinder of caliber 1.

This follows from Perelman's compactness theorem, see 11.7 of [14]: roughly, if this is not true, take a limit of counterexamples. By Perelman's curvature bound (see the last three lines in page 30 of [14]), we see the distance from p to p_1, p_2 goes to ∞ .

Therefore the limit contains a line and splits; it must be the cylinder and that is a contradiction. See also [12], [13], [3] for more details.

Finally, with nonnegative curvature, a singularity must happen everywhere:

Proposition 1.12 (Perelman). *Assume X is a noncollapsing ancient solution of strictly positive curvature. If X develops a singularity at time T , then for all $x \in X$,*

$$(1.13) \quad \lim_{t \rightarrow T^-} R(x, t) = \infty.$$

See the proof of 12.1, Claim 2, in [14].

2. A THEOREM ON ANCIENT SOLUTIONS

Very similar to Definition 1.2, we define

Definition 2.1. *Assume X is a noncollapsing, noncompact ancient solution of strict positive, bounded curvature, defined for time $(-\infty, 0]$. The **caliber** of X at time 0*

$$(2.2) \quad \lim_{x \rightarrow \infty} \frac{1}{R(x, 0)}.$$

In view of Proposition 1.5, the above limit exists; in fact, the solution is close to a cylinder with the above caliber when $x \rightarrow \infty$.

Since $(X, g(0))$ has bounded curvature, by Shi's theorem [16] we can extend the solution for a short time beyond $t = 0$. Moreover, on each time slice, the curvature is bounded, positive; see [17] Theorem 4.14. See also 12.1 of [13] for an alternative argument.

Lemma 2.3. *Assume $g(t)$ is a maximal solution with initial data $(X, g(0))$ that has bounded curvature in each time slice. Then $g(t)$ exists on $[0, C)$, where C is the caliber of $(X, g(0))$. Moreover, if $C < \infty$, then for all $x \in X$, $\lim_{t \rightarrow C^-} R(x, t) = \infty$.*

Here “maximal” means the following. We can give a partial order to the set of all Ricci flow solutions with bounded curvature at each time slice and with initial data $(X, g(0))$: we say $g_1 \leq g_2$ if g_2 is an extension (in time) of g_1 . Maximal means maximal according to this partial order.²

Proof. The proof is exactly the same as the argument in section 2 of [15], where Perelman proved the the life of the *standard solution* is its caliber at initial time. See [12], [13], [3] for more details. \square

Recall that a solution develops a type I singularity, if the solution goes singular at time T and there exists $C \geq 0$ so that

$$(2.4) \quad \limsup_{t \rightarrow T^-} (T - t) \cdot \sup_{x \in X} R(x, t) = C.$$

² Implicitly in this statement, extension(s) may or may not be unique. Recent there have been results on the uniqueness of Ricci flow, these could make some of our arguments more simple.

Theorem 2.5. *There is no noncollapsing, noncompact ancient solution with strictly positive curvature that has a type I singularity.*

We argue by contradiction. Assume there is one such solution X . Notice we assume positive curvature, so the solution is diffeomorphic to \mathbb{R}^3 , its structure is described in Proposition 1.5, see Figure I. In the next two lemmas we show that one can get a global curvature bound similar to (2.4).

Lemma 2.6. *We have*

$$(2.7) \quad \liminf_{t \rightarrow T^-} (T - t) \cdot \sup_x R(x, t) > 0.$$

In particular, $C > 0$ and there exists $c > 0$ so that for all t close to T and all x ,

$$(2.8) \quad c \leq (T - t)R(x, t) \leq C.$$

Proof. By lemma 2.3, at time t the caliber of $(X, g(t))$ is $T - t$. Therefore

$$(2.9) \quad \lim_{x \rightarrow \infty} R(x, t) = \frac{1}{T - t}.$$

We claim there is a universal constant c so that for all $y \in X$,

$$(2.10) \quad R(y, t) \geq c \lim_{x \rightarrow \infty} R(x, t) = \frac{c}{T - t}.$$

As in many occasions in Perelman's papers, e.g. the last paragraph in 11.7 of [14], this follows from Yau's volume comparison argument with base point at infinity. For reader's convenience, we give a sketch:

By doing a rescale we can assume $T - t = 1$. Take a ray γ with $\gamma(0) = y$. If the distance s_1 is sufficiently large, a neighborhood of $\gamma(s_1)$ is close to a (piece of) cylinder with caliber $T - t$. Take $s_2 \gg s_1$, let $p = \gamma(s_2)$, define the one-direction annulus

$$(2.11) \quad A(p, \delta_1, \delta_2) = \{x \in X, |\delta_1 \leq d(x, p) \leq \delta_2, \liminf_{s \rightarrow \infty} d(\gamma(s), x) - (s - s_2) \geq 0\}.$$

The volume comparison based at p implies

$$(2.12) \quad \frac{\text{Vol}(A(p, s_2 - D, s_2 + D))}{\text{Vol}(A(p, s_2 - s_1 - D, s_2 - s_1 + D))} \leq \frac{\int_{s_2 - D}^{s_2 + D} r^2 dr}{\int_{s_2 - s_1 - D}^{s_2 - s_1 + D} r^2 dr} \approx 1,$$

here $D \gg \sqrt{T - t}$ while $D \ll s_1 \ll s_2$. Since $A(p, s_2 - s_1 - D, s_2 - s_1 + D)$, which contains $\gamma(s_1)$, is approximately a cylinder of length $2D$ and caliber $T - t$, we see

$$(2.13) \quad \text{Vol}(A(p, s_2 - s_1 - D, s_2 - s_1 + D)) \sim 2D(T - t) \ll D^3.$$

If $R(y, t)$ is too small, say $R(y, t) \leq c$, then by Perelman's compactness theorem for ancient solution (see the proof of 11.7 of [14]), we have $R \leq 1$ on $B(y, t, D)$, so by the noncollapsing assumption we have

$$(2.14) \quad \text{Vol}(A(p, s_2 - D, s_2 + D)) \geq \text{Vol}(B(y, t, D)) \geq \tau D^3.$$

Now (2.14) together with (2.13) contradicts (2.12). \square

Lemma 2.15. *If X is of type I, then there exists a noncompact, noncollapsing ancient solution with positive curvature that goes singular at time 0, and for all $x \in X_1$ and $t \leq 0$ we have*

$$(2.16) \quad c \leq |t| \cdot R(x, t) \leq C.$$

Proof. Let $t_i \rightarrow T^-$. Take pointed rescale limit at points (x_i, t_i) in the cap region, i.e. at the X_C region (see Proposition 1.5), this will guarantee the limit does not split. Then translate the singular time to 0. The conclusion follows from the previous lemma. \square

However, such ancient solution does not exist:

Theorem 2.17. *There is no noncollapsing, noncompact, ancient solution, with bounded positive curvature at each time slices, satisfying the following:*

- 1). *The solution goes singular at time $T = 0$.*
- 2). *For all (x, t) we have $c \leq |t| \cdot R(x, t) \leq C$.*

We break the proof into a sequence of lemmas. In view of Proposition 1.5, at time t when $x \in X_T$, we say x **is on the cap** at time t ; when $x \in X_T$, we say x **is the center of a tube** at time t ; see Figure I. When curvature is strictly positive, X_C here is diffeomorphic to the solid ball B^3 . A point x may be on the cap at one time and be the center of a tube at another time.

Lemma 2.18. *Let (p_i, t_i) be a sequence of points with $t_i \rightarrow -\infty$ that are on the cap. Then fix a time, say -1 , the points $(p_i, -1)$ goes to space infinity when $i \rightarrow \infty$.*

Proof. In fact, if this is not true, then assume $(p_i, -1) \rightarrow (p, -1)$. Take $(p, -1)$ as the base point and take the constant curve p from inverse time 0 to τ , and compute Perelman's l functional, we get

$$(2.19) \quad l(p, \tau) = \frac{1}{2\sqrt{\tau}} \int_0^\tau \sqrt{s} R ds \leq \frac{1}{2\sqrt{\tau}} \int_0^\tau \sqrt{s} \frac{C}{s+c} ds \leq C, \quad \text{for all } \tau > 0.$$

Thus by Perelman's asymptotic soliton theorem ([14] Proposition 11.2, see also [12], [13]), if we take a rescaled limit at (p, τ) , we shall get a nonflat shrinking soliton. The limit soliton cannot be compact, and since p stay on the cap all the time, the limit does not split, i.e. the limit soliton is of positive curvature. By section 1 of [15], shrinking solitons are either space forms or $S^2 \times \mathbb{R}$, or $S^2 \times \mathbb{R}/\mathbb{Z}_2$, none of them contains a cap-like neighborhood. That is a contradiction. \square

Lemma 2.20. *Given any $C > 0$, there exists ϵ so that for any $\tau > 2$ and for any ancient solutions (M, p) with $\sup R \leq C$ at time -1 , if*

$$(2.21) \quad \tilde{V}(\tau) \leq \tilde{V}(2\tau) + \epsilon,$$

and $l(q, \tau) \leq C$, with the base point of l and \tilde{V} taken at $(p, -1)$, then after rescale by $R(q, \tau)$, a neighborhood of (q, τ) is close to a subset in a shrinking soliton. Here $\tau = -1 - t$ is the inverse time.

Proof. This follows exactly the proof for Proposition 11.2 in [14]. We just need to replace the bound $n/2$ for l there by C . See [18] for more details. \square

Lemma 2.22. *Assume there is an ancient solution X as in Theorem 2.17. Then for any $\epsilon > 0$, there exists $N > 0$, so that for any point $(p, -1)$ that is sufficiently far into space infinity, we have*

$$(2.23) \quad \tilde{V}(\tau) \leq \tilde{V}(2\tau) + \epsilon,$$

for all $\tau > N$. Here we take $(p, -1)$ as the base point for computing l and \tilde{V} .

Proof. Observe, at time t , at space infinity the solution looks like a tube of caliber $|t|$. So no matter where the base point $(p, t) \in X$ for l locates, we have

$$(2.24) \quad \tilde{V}_X(\infty) = \lim_{\tau \rightarrow \infty} \tilde{V}_X(\tau) = \frac{2}{e}(4\pi)^{3/2}.$$

In fact, pick a base point (p, t) , translate the time t to 0; then we have the inverse time τ . Apply Proposition 11.2 in [15], because the asymptotic soliton is noncompact, it must be $S^2 \times \mathbb{R}$ (it is easy to check $S^2 \times \mathbb{R}/\mathbb{Z}_2$ does not happen). Therefore the ancient soliton is a evolving cylinder that goes singular at time 0. On the cylinder we see $\tilde{V}_{\text{cylinder}}(\tau) = \tilde{V}_{\text{cylinder}}(\infty)$ does not depend on τ . Then because $\text{Ric} + \text{Hess}_l = g/(2\tau)$ on the cylinder, it is easy to compute that when $\tau = 1$,

$$(2.25) \quad l_{\text{cylinder}} = 1 + x^2/4,$$

where x is the coordinate in the \mathbb{R} direction. Therefore integrate on the cylinder,

$$(2.26) \quad \tilde{V}_{\text{cylinder}}(\infty) = \tilde{V}_{\text{cylinder}}(1) = \int_{\text{cylinder}} e^{-l} = \frac{2}{e}(4\pi)^{3/2}.$$

By Perelman's growth estimate for l (see Lemma 3.2 in [18]), one can bound the contribution to \tilde{V} outside any give distance. Moreover the scale-down limit of l is just l_{cylinder} . Then one gets

$$(2.27) \quad \tilde{V}_X(\infty) = \tilde{V}_{\text{cylinder}}(\infty) = 2e^{-1}(4\pi)^{3/2}.$$

Especially, pick any $(z, -1)$ in an evolving cylinder Z as the base point for computing l , we get the same limit value $\tilde{V}_Z(\infty) = 2e^{-1}(4\pi)^{3/2}$. Although both \tilde{V}_Z and $\tilde{V}_{\text{cylinder}}$ are computed on evolving cylinder, they are different in that $\tilde{V}_{\text{cylinder}}$ is defined as the limit of reduced volumes over rescaled manifolds, essentially it used the singularity as base point; \tilde{V}_Z used an ordinary point $(z, -1)$ as base point. Therefore $\tilde{V}_{\text{cylinder}}(\tau)$ does not depend on τ while $\tilde{V}_Z(\tau)$ does.

There exists N so that

$$(2.28) \quad \tilde{V}_Z(N) \leq 2e^{-1}(4\pi)^{3/2} + \epsilon/3.$$

Moreover, there is a radius $D \gg \sqrt{N}$ so that

$$(2.29) \quad \tilde{V}_Z(N) - \frac{\epsilon}{3} \leq \int_{B(z, -1-N, D)} N^{-n/2} e^{-l} \leq \tilde{V}_Z(N) \leq \frac{2}{e}(4\pi)^{3/2} + \frac{\epsilon}{3}.$$

If we take pointed limit with base points $(p_i, -1) \in X$ going to space infinity, we get a cylinder of caliber 1. Therefore if we go sufficiently far into the space infinity,

we see the ball B centered at $(p, -1)$ with radius D satisfies that $B \times [-1 - N, -1]$ is close to the corresponding part in space-time of the evolving cylinder Z . So by Perelman's growth estimate on l (see Lemma 3.2 of [18]), we have

$$(2.30) \quad \tilde{V}_X(N) - \frac{\epsilon}{3} \leq \int_{B(p, -N-1, D)} N^{-n/2} e^{-l_X} < \tilde{V}_X(N),$$

$$(2.31) \quad \left| \int_{B(p, -N-1, D)} N^{-n/2} e^{-l_X} - \int_{B(z, -N-1, D)} N^{-n/2} e^{-l_Z} \right| \leq \frac{\epsilon}{3}.$$

Recall \tilde{V} is a decreasing function in τ ; see (7.12)-(7.13) in [14]. So the above combined with (2.24), (2.28) and (2.29) proves the conclusion. \square

Now we prove Theorem 2.17. Take a sequence (p_j, t_j) with $t_j \rightarrow -\infty$ that are on the cap. Then for sufficiently large j with $|t_j| > N$ (here N is the number in Lemma 2.22), we see $(p_j, -1)$ is so far into space infinity that if we use $(p_j, -1)$ as the base point for l , we get

$$(2.32) \quad \tilde{V}(\tau) \leq \tilde{V}(2\tau) + \epsilon;$$

for all $\tau > N$. In particular we take $(p_j, -1)$ as the base point and use $\tau = -1 - t_j$. The type I assumption implies a bound of l at (p_j, t_j) , evaluated on the constant curve p_j . Therefore by Lemma 2.20, after rescale by $R(p_i, t_i)$ we see a big neighborhood of (p_i, t_i) is close to a set in a shrinking soliton. But this is impossible because $(x_i, t_i) \in X_C$. \square

3. FORWARD LIMIT AND ETERNAL SOLUTIONS

To get more concrete information, especially for taking forward limit, we need the following lemma, which could be used as an alternative definition of degenerate singularity; this is very similar to Lemma 4.3 in Perelman's second paper [15].

Lemma 3.1. *Assume a degenerate singularity happens at time T , and the solution does not extinct everywhere at T . Then there exists a positive real number $\epsilon > 0$ (depending on the solution) and a compact set $B \subset M$ that is diffeomorphic to a solid 3-ball, so that there exists a neighborhood of ∂B , for all $t \in [T - \epsilon, T)$, after rescale the metric by $R(p, t)$, is 10^{-2} -close to a cylinder of scalar curvature 1 and length 100; and we have*

$$(3.2) \quad \limsup_{t \rightarrow T^-} \sup_{p \in \partial B} R(p, t) < \infty.$$

Moreover, let $p(t) \in B$ be a point so that

$$(3.3) \quad d_{g(t)}(p(t), \partial B) = \sup_{p \in B} d_{g(t)}(p, \partial B),$$

then

$$(3.4) \quad \lim_{t \rightarrow T^-} R(p(t), t) = \infty.$$

Proof. Let (x_i, t_i) be a sequence of points as in Definition 1.7. Since M is compact, we can assume $x_i \rightarrow x_\infty$. Observe

$$(3.5) \quad \lim_{t \rightarrow T^-} R(x_\infty, t) = \infty.$$

In fact, if this is not true, then for some $C > 0$, there exists t so that $T - t$ is arbitrarily small while $R(x, t) < C$; we pick such a time t^* that is sufficiently close to T . By the gradient estimate (1.9) in the space direction, we see a neighborhood \mathcal{N} of (x_∞, t^*) has bounded curvature. Now because we have chosen t^* sufficiently close to T , by the time-direction gradient estimate in (1.9), \mathcal{N} has a uniform curvature bound on the time interval (t^*, T) . So on \mathcal{N} , the metrics from time t^* to T are uniformly equivalent. In particular, for sufficiently large i we have $x_i \in \mathcal{N}$; this contradicts to the fact $R(x_i, t_i) \rightarrow \infty$.

It is assumed that the solution does not extinct everywhere at time T . As we have seen above, by the gradient estimate (1.9), near time T , curvature cannot increase/decrease suddenly. So there is a point y so that

$$(3.6) \quad \limsup_{t \rightarrow T^-} R(y, t) = C < \infty.$$

Pick a time t^* that is sufficiently to T so that

$$(3.7) \quad R(x_\infty, t^*) > \eta^3 \cdot \max\{r_0^{-2}, C\},$$

here r_0 is the canonical neighborhood parameter in Proposition 1.4, and η is the constant in Perelman's splitting argument, see Proposition 1.10. Take a minimal geodesic γ with respect to $g(t^*)$, from x_∞ to y , parametrized by s . Let r_1 be so that $r_1^{-2} = \eta \cdot \max\{r_0^{-2}, C\}$, and

$$(3.8) \quad s_0 = \inf\{s | R(\gamma(s), t^*) = r_1^{-2}\}, \quad s_1 = \inf\{s | R(\gamma(s), t^*) = 4r_1^{-2}\}.$$

Now since t^* is sufficiently close to T , by applying the gradient estimate (1.9), we can assume on the time interval (t^*, T) we have

$$(3.9) \quad R(\gamma(s_0), t) \leq 2r_1^{-2}.$$

By Proposition 1.10, the canonical neighborhood of $\gamma(s_0)$ at time t^* is a tube, with radius $r_1\sqrt{2}$. We now remove a center sphere S^2 in this tube at $\gamma(s_0)$ from the manifold $(M, g(t^*))$ and get an incomplete Riemannian manifold, possibly with 2 components. In any case, pick the component that contains x_∞ and take its closure M^* . ∂M^* is either connected or has two components, all diffeomorphic to S^2 . Write $S = \partial M^*$ when ∂M^* is connected; when ∂M^* has two components, let S the component of ∂M^* that is closer to $\gamma(s_0 - \delta)$, with $\delta = R(\gamma(s_0), t^*)^{-1/2}$. In another word, S is directly hit by the part of γ coming from x_∞ .

The Ricci flow still runs on this manifold (it runs on the original manifold M , we just remove a sphere). For all $t \in [t^*, T)$, pick a point $q(t)$ that is furthest from S :

$$(3.10) \quad d_t(q(t), S) = \max\{d(x, S) | x \in M^*\}.$$

Let α be a minimum $g(t)$ -geodesic from S^2 to $q(t)$. At time t^* , S is the start of a piece of tube, in which γ is passing through, with length $|s_1 - s_0|$, along which the scalar curvature changes from r_1^{-2} to $4r_1^{-2}$. There are two cases:

Case I. For all s with

$$s \geq |s_1 - s_0|,$$

we have

$$(3.11) \quad R(\alpha(s), t^*) \geq 2r_1^{-2}.$$

This means, if ∂M^* has another component S' , then the canonical neighborhoods of $\alpha(s)$ does not intersect S' ; in fact a neighborhood of S' is very close to a long tube with scalar curvature r_1^{-2} . By the classification of canonical neighborhood and Proposition 1.10, M^* is being covered by canonical neighborhoods and therefore $\partial M^* = S$ is connected. Notice the neighborhood near $q(t^*)$ is necessarily a cap.

In particular, M^* is diffeomorphic to a solid ball or $RP^3 - B$. Now $x_\infty \in M^*$, and the curvature at $\partial M^* = S^2$ remain bounded when $t \rightarrow T$, we conclude that for i sufficiently large, $x_i \in M^*$. In particular, the possibility $M^* = RP^3 - B$ is ruled out, because M^* contains a cap diffeomorphic to B^3 . Therefore, we can just make $p(t) = q(t)$ and $B = M^*$.

Case II.

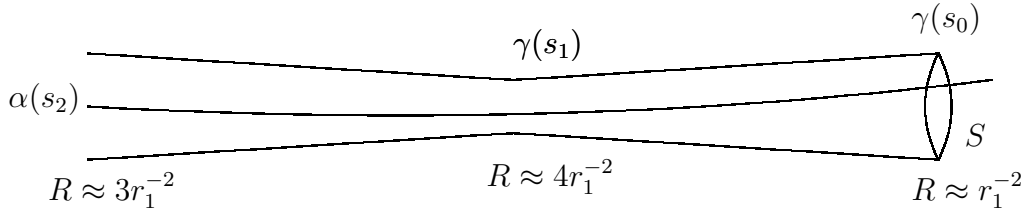


Figure III.

There is a minimal s' with $s > |s_0 - s_1|$ so that

$$(3.12) \quad R(\alpha(s'), t^*) = 2r_1^{-2}.$$

We will prove that this is impossible. Assume it does happen, we will find a minimum s_2 with $s_2 > |s_0 - s_1|$ so that

$$(3.13) \quad R(\alpha(s_2), t^*) = 3r_1^{-2}.$$

Since we choose t^* to be sufficiently close to T , by the gradient estimate (1.9), a neighborhood \mathcal{N}_2 of $\alpha(s_2)$ has a curvature bound

$$(3.14) \quad R \leq 10r_1^{-2}$$

for all time $t \in [t^*, T)$. Again by Proposition 1.10, the canonical neighborhood near \mathcal{N}_2 is a cylinder. Now we cut M^* at a center sphere at $\alpha(s_2)$ and take the component \mathcal{C} that contains $\gamma(s_1)$. This component \mathcal{C} is just a piece of a cylinder, see also Proposition 1.10. In particular, by the choice of $\gamma(s_1)$, we have $x_\infty \in \mathcal{C}$. Now we have uniform curvature bounds on the two ends of this cylinder on the time interval $[t^*, T)$. In particular, for sufficient large i , we have $x_i \in \mathcal{C}$. On the other hand, also by (1.9), there is actually a lower bound

$$(3.15) \quad \frac{1}{10}r_1^{-2} \leq R(x, t)$$

for all $x \in \mathcal{C}$ and $t \in [t^*, T)$. This means for all time $t \in [t^*, T)$, all $x \in \mathcal{C}$ has a canonical neighborhood. We already know that near the two ends, the canonical neighborhoods are both cylinders, therefore we conclude all points in \mathcal{C} has cylinder as a canonical neighborhood. This contradicts the fact that the canonical neighborhood of (x_i, t_i) is a cap. \square

Assume a degenerate singularity happens at time T . By taking a rescale limit at $(p(t), t)$ above, with $t \rightarrow T^-$, we get an ancient solution X . Since the canonical neighborhood of $p(t)$ are of case iv) in Proposition 1.4, the limit X is necessarily of strictly positive curvature, otherwise the solution is $S^2 \times \mathbb{R}$ or its \mathbb{Z}_2 quotient, by the strong maximum principle. By a translation in time, X exists on $(-\infty, 0]$.

Let $T_1 > 0$ be the maximum time so that there is an extension of X up to time T_1 and for all $t < T_1$, the curvature of $g(t)$ is positive and bounded. Therefore T_1 is the caliber of $(X, g(0))$, see Lemma 2.3. We are not saying this extension is unique. Moreover, *a priori*, when $0 < T' < T_1$ we do not know if the solution $(X, g(t))$, with $-\infty < t \leq T'$, is a rescale limit of the original Ricci flow on M .

Lemma 3.16. *There exists an extension $(X, g(t))$ with $-\infty < t < T_1$, where T_1 is the caliber of $(X, g(0))$, so that for all $T' > 0$ with $T' < T_1$, the ancient solution X over time $(-\infty, T']$ is a rescale limit of the original solution on M .*

Proof. Consider the set of time T , so that there is an extension of X to time T , and the ancient solution X over time $(-\infty, T)$ is a rescale limit of the original solution on M . Let T_0 be the supremum of this set.

First of all, we have $T_0 > 0$. In fact, a rescale of M is close to $(X, g(0))$ under the pointed C^3 topology. We can assume the rescale is taken at $p(t)$ in Lemma 3.1, remember that the original solution does not go singular unless $R(p(t), t)$ goes to infinity. Now the corresponding neighborhood $\mathcal{N}(t)$ of $p(t)$ contains a cap part \mathcal{N}_C and a tube part \mathcal{N}_T that corresponds to the X_C, X_T decomposition of X .

From the definition of $p(t)$ in (3.3), as long as the Ricci flow is running (at least for a short time when the metric geometry of \mathcal{N} does not change too much), $p(t + \Delta t)$, the global maximum point of a distance function, must remain in \mathcal{N} . The gradient estimate (1.9) implies the curvature on \mathcal{N} cannot go to infinity immediately; so $R(p(t + \Delta t), t + \Delta t)$ does not go to infinity immediately. During this time period the solution cannot go singular. Therefore we can take a *definite* short time forward limit at $p(t)$. This limit extends X , so $T_0 > 0$.

Now we can repeat this argument (i.e. view T_0 as time “0”) as long as X does not go singular, therefore we must have $T_0 = T_1$, the caliber of $(X, g(0))$. \square

We would like to have $T_1 = \infty$, that is, X is an eternal solution by Lemma 2.3. Assume this is not the case, then it is necessary that the solution develops a singularity at time T_1 . By Theorem 2.5, this singularity is of type II as defined in [10], that is,

$$(3.17) \quad \limsup_{t \rightarrow T^-} (T_1 - t) \cdot \sup_x R(x, t) = \infty.$$

In all cases, by taking a further limit we will get an eternal solution in which scalar curvature reaches maximum. This *point-picking* procedure is well known; [14], [15], [12], [5]. We give a quick account here.

We will find a sequence of points (p_n, t_n) , so that when we rescale the solution by $R(x_n, t_n)$ (i.e. first make $R(x_n, t_n) = 1$), and then translate the time t_n into 0 (i.e. our base point is now $(p_n, 0)$ with $R(p_n, 0) = 1$), then the solution exists on the interval $(-\infty, n]$, moreover,

$$(3.18) \quad 1 = R(p_n, 0) \geq \left(1 - \frac{1}{n+1}\right) \sup\{R(x, t) \mid x \in X, t \leq n\}.$$

In fact, start with $(p, 0) = (p^{(0)}, t^{(0)})$ so that $R(p, 0)$ is maximal at time 0, and

$$(3.19) \quad (T_1 - t^{(0)})R(p^{(0)}, t^{(0)}) > (n+1)^2$$

if X is not eternal. Assume there is a point $p^{(1)}$ and time $t^{(1)}$ so that

$$(3.20) \quad R(p^{(1)}, t^{(1)}) > \left(1 + \frac{1}{n}\right)R(p^{(0)}, t^{(0)}), \quad t^{(1)} - t^{(0)} \leq nR(p^{(0)}, t^{(0)})^{-1}.$$

Without lose of generality we assume $R(p^{(1)}, t^{(1)})$ is almost maximal at the time slice $t = t^{(1)}$. If $(p^{(1)}, t^{(1)})$ does not satisfy our requirement, we continue to find $(p^{(2)}, t^{(2)})$... In general,

$$(3.21) \quad R(p^{(k)}, t^{(k)}) > \left(1 + \frac{1}{n}\right)^k R(p^{(0)}, t^{(0)}), \quad t^{(k)} - t^{(0)} \leq n(n+1)R(p^{(0)}, t^{(0)})^{-1}.$$

This is a contradiction, since the solution X exists up to time $n(n+1)R(p^{(0)}, t^{(0)})^{-1}$, therefore there is a curvature bound at the time slice $t = n(n+1)R(p^{(0)}, t^{(0)})^{-1}$ (see also Theorem 11.4 in [14]). By Hamilton's Harnack inequality [8], this scalar curvature bound actually holds for all $t \leq n(n-1)R(p^{(0)}, t^{(0)})^{-1}$.

Now take a pointed limit Z of $(X, (p_n, t_n), R(p_n, t_n)g)$, we get an eternal solution on which the maximal of scalar curvature is reached. Therefore we can apply Hamilton's theorem on eternal solutions (see [9]) to conclude that the limit Z is a steady gradient soliton. Now by Lemma 3.16 we finally get

Theorem 3.22. *Assume $g(t)$ develops a degenerate singularity at time T . Then then a rescale limit of $g(t)$ towards time T is a steady gradient soliton.*

Proof. We already proved the first case, i.e. the solution does not extinct everywhere at time T .

The second case, that M extincts everywhere at time T , is actually easier. In fact the solution does not extinct unless the everywhere the curvature goes to infinity; therefore there is no difficulty in obtaining forward time limit. On the other hand, when we take limit as before and get an ancient solution X with nonnegative curvature, then X is noncompact. In fact if this is not true, then X is diffeomorphic to S^3 or PR^3 ; by taking a forward limit we see that the *original* solution must be of strictly positive curvature near time T , in particular it will get rounder and rounder and cannot be a degenerated singularity. So the proof goes like the first case. \square

Corollary 3.23. *A rescale limit of Perelman's standard solution is the Bryant soliton.*

Proof. This follows from Theorem 2.5. Notice that the soliton we get is also radial symmetric; see [15], chapter 12 of [13]. Therefore by the discussion in [5] the soliton must be the Bryant soliton. \square

Perelman conjectured that Bryant soliton is the only noncompact ancient solution with positive curvature; this will imply all the results in our paper. It should be true that every rescale limit taken along $(p(t), t)$ in Lemma 3.1 must be the Bryant soliton. Unfortunately currently there are several difficulties in this direction.

Finally we mention that analogue result of Lemma 3.1 for $RP^3 - B^3$ caps also holds. In fact one can prove that locally the singularity is of type I and a rescale limit is $S^2 \times \mathbb{R}/\mathbb{Z}_2$; one can get an example for such a singularity (e.g. taking \mathbb{Z}_2 quotient of the examples in chapter 2, [4]). Therefore we will not classify this case as degenerate.

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